Majorization and entropy at the output of bosonic Gaussian channels

Andrea Mari

NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, I-56127 Pisa, Italy

V. Giovannetti
(SNS, Pisa)

A. S. Holevo
(Steklov Math. Inst., Moscow)

R. Garcìa-Patròn
(QuIC, Bruxelles and MPI, Garching)

N. J. Cerf
(QuIC, Bruxelles)

Outline

Long introduction

- General question behind this talk
- Overview of recent breakthroughs on the field
- Gaussian states and Gaussian channels
- An old result about the quantum beam splitter

Proof of the result

- Proof of the majorization conjecture
- Implications and outlook

Outline

Long introduction

- **General question behind this talk**
- Overview of recent breakthroughs on the field
- Gaussian states and Gaussian channels
- An old result about the quantum beam splitter

Proof of the result

- Proof of the majorization conjecture
- Implications and outlook

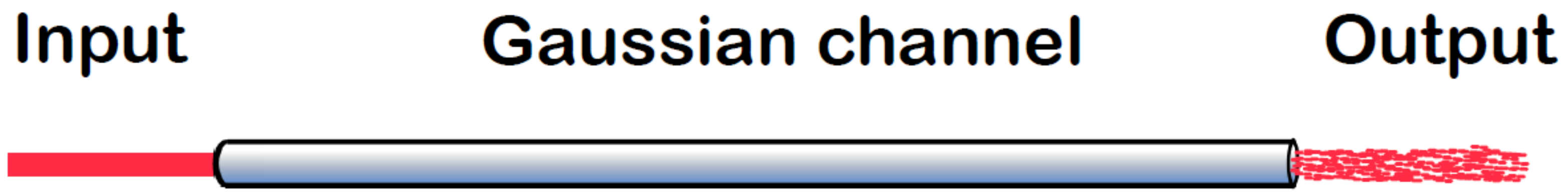
Input **Gaussian channel** **Output**

Radiation modes:
optical waves,
microwaves,
radio waves,
etc.

Physical transmission line:
optical fiber, free space communication,
satellite link, micro-waveguide, etc.

Noisy output signal

WARNING for typical QIP attendees:
this talk may have some implications in the real world.



Radiation modes:
optical waves,
microwaves,
radio waves,
etc.

Physical transmission line:
optical fiber, free space communication,
satellite link, micro-waveguide, etc.

Noisy output signal

General question behind this talk:

“ What is the minimum “noise” or “disorder” achievable at the output state, optimizing over all possible input states? ”

Important implications in communication theory:
what are the code-words which are less disturbed by the channel ?

What does it mean that ρ_1 is more "disordered" then ρ_2 ?

(i) Von Neumann entropy criterion

$$S(\rho_1) \equiv -\mathrm{Tr}\ \rho_1 \log(\rho_1) \ > S(\rho_2) \equiv -\mathrm{Tr}\ \rho_2 \log(\rho_2)$$

(ii) Quantum state majorization criterion

$$\rho_2 \succ \rho_1$$

$$\sum_{j=1}^{k} \lambda_j^{\rho_1} \leq \sum_{j=1}^{k} \lambda_j^{\rho_2}, \quad \forall k \geq 1,$$

$\{\lambda_1^{\rho_1}, \lambda_2^{\rho_1}, \ldots\}$
$\{\lambda_1^{\rho_2}, \lambda_2^{\rho_2}, \ldots\}$ = eigenvalues of ρ_1 and ρ_2 arranged in decreasing order

Remark: condition (ii) is very strong, indeed it can be shown that,

$$\rho_2 \succ \rho_1 \Leftrightarrow \mathrm{Tr} f(\rho_1) \geq \mathrm{Tr} f(\rho_2)$$

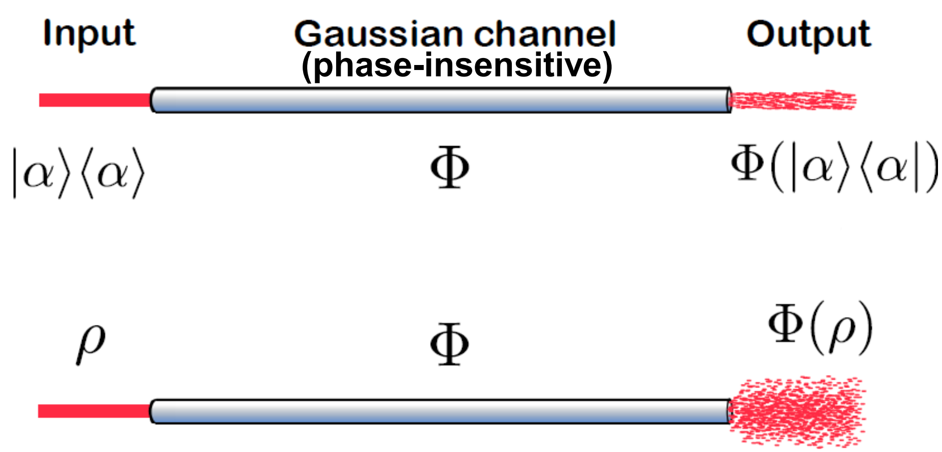
for every concave function f

In particular, $\rho_2 \succ \rho_1 \Rightarrow S(\rho_1) \geq S(\rho_2)$

Coherent states:

$$|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle \qquad \alpha \in \mathbb{C}$$

Vacuum state



(i) Minimum output entropy conjecture: *the output entropy is minimized by coherent input states*

$$S(\Phi(\rho)) \geq S(\Phi(|\alpha\rangle\langle\alpha|)), \quad \forall \rho, |\alpha\rangle$$

(ii) Majorization conjecture: *output of coherent input states majorize all other output states*

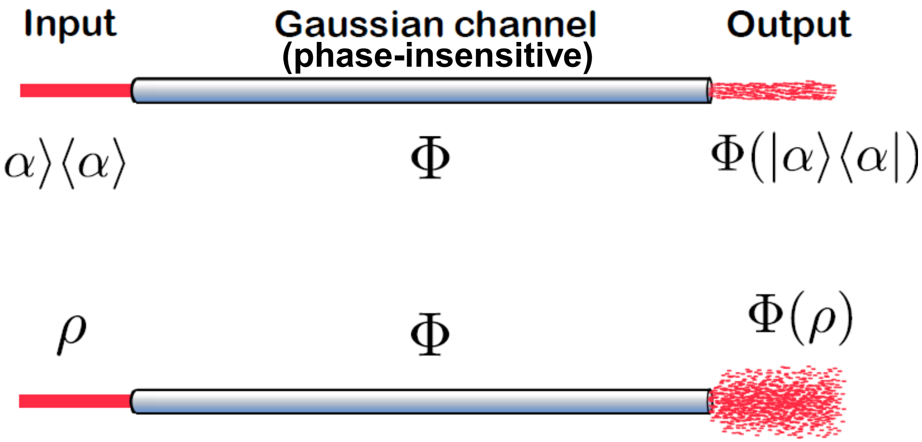
$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \quad \forall \rho, |\alpha\rangle$$

Giovannetti, *et al.*, PRA, (2004)
Holevo, Werner, PRA, (1997)

Coherent states: $\alpha \in \mathbb{C}$

$$|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle$$

Vacuum state



(i) Minimum output entropy conjecture: *the output entropy is minimized by coherent input states*

$$S(\Phi(\rho)) \geq S(\Phi(|\alpha\rangle\langle\alpha|)), \quad \forall \rho, |\alpha\rangle$$

(ii) Majorization conjecture: *output of coherent input states majorize all other output states*

$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \quad \forall \rho, |\alpha\rangle$$

PROOF GIVEN IN THIS TALK

Giovannetti, *et al.*, PRA, (2004)
Holevo, Werner, PRA, (1997)

Outline

Long introduction

- General question behind this talk
- **Overview of recent breakthroughs on the field**
- Gaussian states and Gaussian channels
- An old result about the quantum beam splitter

Proof of the result

- Proof of the majorization conjecture
- Implications and outlook

A hierarchy of conjectures (situation before 8/12/2013)

Majorization conjecture

"the output of coherent input states majorize all other output states"

Giovannetti, *et al.*, PRA, (2004)

⇓

Minimum output entropy conjecture

"*the output entropy is minimized by coherent input states*"

Giovannetti, *et al.*, PRA, (2004)

⇓

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The output Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

⇒

CLASSICAL CAPACITY
(explicit formula for phase-insensitive Gaussian channels)

A hierarchy of conjectures (situation before 8/12/2013)

Majorization conjecture

"the output of coherent input states majorize all other output states"

Giovannetti, *et al.*, PRA, (2004)

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Minimum output entropy conjecture

"the output entropy is minimized by coherent input states"

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

König, Smith, Nat. Photon, (2013)

bounds for classical capacity

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The output Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

CLASSICAL CAPACITY
(explicit formula for phase-insensitive Gaussian channels)

A hierarchy of conjectures (situation before 8/12/2013)

Majorization conjecture

"the output of coherent input states majorize all other output states"

Giovannetti, *et al.*, PRA, (2004)

Phase-space majorization conjecture

"the Q-function of a coherent states majorizes every other Q-function"

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Minimum output entropy conjecture

"the output entropy is minimized by coherent input states"

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

König, Smith, Nat. Photon, (2013)

bounds for classical capacity

Additivity and minimum of output Rényi entropies conjecture

"*output Rényi entropies of every order p>1 are additive and minimized by coherent input states*"

Giovannetti, *et al.*, PRA, (2004)

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The output Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan *et al.*, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY
(explicit formula for phase-insensitive Gaussian channels)

A hierarchy of conjectures (situation before 8/12/2013)

Majorization conjecture

“the outp *tates*
majorize *s”*

RA, (2004)

Phase-space majorization conjecture

“the Q-function of a coherent states majorizes every other Q-function”

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen,
Proc. Inf. Th., IEEE, (2008).

Minimum output entropy conjecture

“the output entropy is minimized by coherent input states”

Giovannetti, *et al.*, PRA, (2004)

Additivity and minimum of output Rényi entropies conjecture

“output Rényi entropies of every order p>1 are additive and minimized by coherent input states”

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

König, Smith, Nat. Photon, (2013)

bounds for classical capacity

Gaussian optimal encoding conjecture

“the max of classical capacity is obtained by Gaussian input states”

Gaussian additivity conjecture

“The output Holevo information is additive (entangled code-words are not required)”

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan *et al.*, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY
(explicit formula for phase-insensitive Gaussian channels)

A hierarchy of conjectures (situation before 8/12/2013)

Majorization conjecture

"the outp... tates majorize... s"

...RA, (2004)

Phase-space majorization conjecture

"the Q-function of a coherent states majorizes every other Q-function"

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Minimum outp... onjecture

"the output e... imized by coherent ..."

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

König, Smith, Nat. Photon, (2013)

bounds for classical capacity

Additivity and minimum of output Rényi entropies conjecture

"*output Rényi entropies of every order p>1 are additive and minimized by coherent input states*"

Giovannetti, *et al.*, PRA, (2004)

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The output Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan *et al.*, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY (explicit formula for phase-insensitive Gaussian channels)

A hierarchy of conjectures (situation before 8/12/2013)

Majorization conjecture

“the output of coherent input states majorize all other output states”

Giovannetti, *et al.*, PRA, (2004)

Phase-space majorization conjecture

“the Q-function of a coherent states majorizes every other Q-function”

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Entropy power inequality conjecture

König, Smith, Nat. Photon, (2013)

bounds for classical capacity

Additivity and minimum of output Rényi entropies conjecture

“*output Rényi entropies of every order p>1 are additive and minimized by coherent input states*”

Giovannetti, *et al.*, PRA, (2004)

Gaussian optimal encoding conjecture

“the max of classical capacity is obtained by Gaussian input states”

Gaussian additivity conjecture

“The output Holevo information is additive (entangled code-words are not required)”

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan *et al.*, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY (explicit formula for phase-insensitive Gaussian channels)

A hierarchy of ~~conjectures~~ proofs (8/12/2013)

Majorization conjecture

"the output of coherent input states majorize all other output states"

Giovannetti, *et al.*, PRA, (2004)

Phase-space majorization conjecture

"the Q-function of a coherent states majorizes every other Q-function"

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Minimum output entropy conjecture

"the output entropy is minimized by coherent input states"

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

König, Smith, Nat. Photon, (2013)

Additivity and minimum of output Rényi entropies conjecture

"output Rényi entropies of every order p>1 are additive and minimized by coherent input states"

Giovannetti, *et al.*, PRA, (2004)

bounds for classical capacity

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan *et al.*, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY

(explicit formula for phase-insensitive Gaussian channels)

A hierarchy of ~~conjectures~~ proofs (12/12/2013)

Majorization conjecture

"the output of coherent input states majorize all other output states"

Mari, Giovannetti, Holevo, arXiv:1312.6225 Nat. Comm. (2014)

Giovannetti, *et al.*, PRA, (2004)

Phase-space majorization conjecture

"the Q-function of a coherent states majorizes every other Q-function"

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Minimum output entropy conjecture

"the output entropy is minimized by coherent input states"

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

König, Smith, Nat. Photon, (2013)

Additivity and minimum of output Rényi entropies conjecture

"*output Rényi entropies of every order p>1 are additive and minimized by coherent input states*"

Giovannetti, *et al.*, PRA, (2004)

bounds for classical capacity

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan *et al.*, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY (explicit formula for phase-insensitive Gaussian channels)

A hierarchy of ~~conjectures~~ proofs (21/12/2013)

Phase-space majorization conjecture

“the Q-function of a coherent states majorizes every other Q-function”

Lieb, Solovej, arXiv:1208.3632

Majorization conjecture

“the output of coherent input states majorize all other output states”

Mari, Giovannetti, Holevo, arXiv:1312.6225 Nat. Comm. (2014)

Giovannetti, *et al.*, PRA, (2004)

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Additivity and minimum of output Rényi entropies conjecture

“output Rényi entropies of every order p>1 are additive and minimized by coherent input states”

Giovannetti, *et al.*, PRA, (2004)

Minimum output entropy conjecture

“the output entropy is minimized by coherent input states”

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

König, Smith, Nat. Photon, (2013)

bounds for classical capacity

Strong converse theorem for classical capacity of Gaussian channels

Bardhan *et al.*, IEEE Inf. Th., (2014)

Gaussian optimal encoding conjecture

“the max of classical capacity is obtained by Gaussian input states”

Gaussian additivity conjecture

“The Holevo information is additive (entangled code-words are not required)”

Holevo, Werner, PRA, (1997)

CLASSICAL CAPACITY
(explicit formula for phase-insensitive Gaussian channels)

Giovannetti, Garcia-Patron, Cerf, Holevo arXiv:1312.6225, Nat. Phot. (2014)

A hierarchy of ~~conjectures~~ proofs (16/01/2014)

Phase-space majorization conjecture

"the Q-function of a coherent states majorizes every other Q-function"

Lieb, Solovej, arXiv:1208.3632

Majorization conjecture

"the output of coherent input states majorize all other output states"

Mari, Giovannetti, Holevo, arXiv:1312.6225 Nat. Comm. (2014)

Giovannetti, *et al.*, PRA, (2004)

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Additivity and minimum of output Rényi entropies conjecture

"output Rényi entropies of every order p>1 are additive and minimized by coherent input states"

Giovannetti, *et al.*, PRA, (2004)

Minimum output entropy conjecture

"the output entropy is minimized by coherent input states"

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

König, Smith, Nat. Photon, (2013)

bounds for classical capacity

Strong converse theorem for classical capacity of Gaussian channels

Bardhan, Garcia-Patron, Wilde, Winter arXiv:1401.4161, IEEE Inf. Th., (2014)

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

CLASSICAL CAPACITY (explicit formula for phase-insensitive Gaussian channels)

Giovannetti, Garcia-Patron, Cerf, Holevo arXiv:1312.6225, Nat. Phot. (2014)

A hierarchy of ~~conjectures~~ proofs (03/02/2014)

Majorization conjecture

"the output of coherent input states majorize all other output states"

Mari, Giovannetti, Holevo, arXiv:1312.6225 Nat. Comm. (2014)

Giovannetti, *et al.*, PRA, (2004)

Phase-space majorization conjecture

"the Q-function of a coherent states majorizes every other Q-function"

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Minimum output entropy conjecture

"the output entropy is minimized by coherent input states"

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

De Palma, Mari, Giovannetti, arXiv: 1402.0404, Nat. Phot. (2014)

König, Smith, Nat. Photon, (2013)

De Palma, Mari, Lloyd, Giovannetti, arXiv:1408.6410

bounds for classical capacity

Additivity and minimum of output Rényi entropies conjecture

"output Rényi entropies of every order p>1 are additive and minimized by coherent input states"

Giovannetti, *et al.*, PRA, (2004)

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan, Garcia-Patron, Wilde, Winter arXiv:1401.4161, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY
(explicit formula for phase-insensitive Gaussian channels)

Giovannetti, Garcia-Patron, Cerf, Holevo arXiv:1312.6225, Nat. Phot. (2014)

A hierarchy of ~~conjectures~~ proofs (16/05/2014)

Majorization conjecture

"the output of coherent input states majorize all other output states"

Mari, Giovannetti, Holevo, arXiv:1312.6225 Nat. Comm. (2014)

Giovannetti, *et al.*, PRA, (2004)

Phase-space majorization conjecture

"the Q-function of a coherent states majorizes every other Q-function"

V. Giovannetti, A. S. Holevo, A. Mari arXiv:1405.4066

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Minimum output entropy conjecture

"the output entropy is minimized by coherent input states"

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Giovannetti, *et al.*, PRA, (2004)

Additivity and minimum of output Rényi entropies conjecture

"output Rényi entropies of every order p>1 are additive and minimized by coherent input states"

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

De Palma, Mari, Giovannetti, arXiv: 1402.0404, Nat. Phot. (2014)

König, Smith, Nat. Photon, (2013)

De Palma, Mari, Lloyd, Giovannetti, arXiv:1408.6410

bounds for classical capacity

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan, Garcia-Patron, Wilde, Winter arXiv:1401.4161, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY
(explicit formula for phase-insensitive Gaussian channels)

Giovannetti, Garcia-Patron, Cerf, Holevo arXiv:1312.6225, Nat. Phot. (2014)

A hierarchy of ~~conjectures~~ proofs (state of the art)

Majorization conjecture

"the output of coherent input states majorize all other output states"

Mari, Giovannetti, Holevo, arXiv:1312.6225 Nat. Comm. (2014)

Giovannetti, *et al.*, PRA, (2004)

Phase-space majorization conjecture

"the Q-function of a coherent states majorizes every other Q-function"

V. Giovannetti, A. S. Holevo, A. Mari arXiv:1405.4066

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Minimum output entropy conjecture

"the output entropy is minimized by coherent input states"

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

De Palma, Mari, Giovannetti, arXiv: 1402.0404, Nat. Phot. (2014)

König, Smith, Nat. Photon, (2013)

De Palma, Mari, Lloyd, Giovannetti, arXiv:1408.6410

bounds for classical capacity

Additivity and minimum of output Rényi entropies conjecture

"output Rényi entropies of every order p>1 are additive and minimized by coherent input states"

Giovannetti, *et al.*, PRA, (2004)

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan, Garcia-Patron, Wilde, Winter arXiv:1401.4161, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY (explicit formula for phase-insensitive Gaussian channels)

Giovannetti, Garcia-Patron, Cerf, Holevo arXiv:1312.6225, Nat. Phot. (2014)

Outline

Long introduction

- General question behind this talk
- Overview of recent breakthroughs on the field
- **Gaussian states and Gaussian channels**
- An old result about the quantum beam splitter

Proof of the result

- Proof of the majorization conjecture
- Implications and outlook

Quantum bosonic systems

e.g. single mode of electromagnetic radiation of frequency ω → $H = \hbar\omega a^\dagger a$, $[a, a^\dagger] = 1$

Eigenstates (Fock states): $a^\dagger a|n\rangle = n|n\rangle$

$$|0\rangle, |1\rangle, \ldots |n\rangle, \ldots$$

vacuum

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

Coherent states: $a|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C}$ → $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{n!}|n\rangle$

$$|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle$$

Displacement (or Weyl) operator $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$

Phase space quasi-distributions

Characteristic function (or Fourier-Weyl transform):

$$\chi(\mu) = \mathrm{Tr}[e^{\mu a^\dagger - \mu^* a}\rho] = \mathrm{Tr}[D(\mu)\rho], \quad \mu \in \mathbb{C}$$

Wigner function: $$W(\alpha) = \frac{1}{\pi^2}\int e^{\mu^*\alpha - \mu\alpha^*}\chi(\mu)d^2\mu, \quad \alpha \in \mathbb{C}$$

(closest analogue to a classical phase-space density but can assume **negative values**)

Q-function: $$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle, \quad \alpha \in \mathbb{C}$$

("not so close analogue" to a phase-space distribution but at least it is **positive**)

One can use it to define entropy (and majorization) as for classical random variables

Wherl entropy: $$S_W = -\int Q(\alpha)\log[Q(\alpha)]d^2\alpha$$

Gaussian states

Gaussian states = states with Gaussian phase space quasi-distribution

vacuum state	coherent states	thermal state
$\rho = \vert 0\rangle\langle 0\vert$	$\rho = \vert\alpha\rangle\langle\alpha\vert$	$\rho = \frac{e^{-\beta\hbar\omega a^\dagger a}}{\mathcal{Z}}$ $N = \langle a^\dagger a\rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$
$\chi(\mu) = e^{-\frac{\vert\mu\vert^2}{2}}$	$\chi(\mu) = e^{-\frac{\vert\mu\vert^2}{2} + \mu^*\alpha - \mu\alpha^*}$	$\chi(\mu) = e^{-(N+\frac{1}{2})\vert\mu\vert^2}$
$W(\alpha') = \frac{2}{\pi} e^{-2\vert\alpha'\vert^2}$	$W(\alpha') = \frac{2}{\pi} e^{-2\vert\alpha'-\alpha\vert^2}$	$W(\alpha') = \frac{1}{\pi\left(\frac{1}{2} + N\right)} e^{-\frac{\vert\alpha'\vert^2}{\frac{1}{2}+N}}$
$Q(\alpha') = \frac{1}{\pi} e^{-\vert\alpha'\vert^2}$	$Q(\alpha') = \frac{1}{\pi} e^{-\vert\alpha'-\alpha\vert^2}$	$Q(\alpha') = \frac{1}{\pi(1+N)} e^{-\frac{\vert\alpha'\vert^2}{1+N}}$

Gaussian channels

Gaussian channels = CPT operations mapping Gaussian states into Gaussian states

Gaussian unitary channels: $U = e^{iH}, \quad H = H^\dagger =$ quadratic in $a, a^\dagger$

In general: $\Phi(\rho) = \mathrm{Tr}_{\mathrm{B}}[U(\rho \otimes \rho_B)U^\dagger]$

Gaussian

Gaussian channels

Gaussian channels = CPT operations mapping Gaussian states into Gaussian states.

Gaussian unitary channels: $U = e^{iH}, \quad H = H^\dagger =$ quadratic in $a, a^\dagger$

In general: $\Phi(\rho) = \mathrm{Tr_B}[U(\rho \otimes \rho_B)U^\dagger]$

Gaussian

Phase-insensitive Gaussian channels = those commuting with phase space rotations

(gauge-covariant) $\Phi(e^{-i\theta a^\dagger a}\rho e^{i\theta a^\dagger a}) = e^{-i\theta a^\dagger a}\Phi(\rho)e^{i\theta a^\dagger a}$

For a single mode,

$$\chi(\rho) \xrightarrow{\Phi} \chi(\lambda_1 \ \mu)e^{-\lambda_2 \frac{|\mu|^2}{2}}, \quad \lambda_2 \geq |1-\lambda_1|^2$$

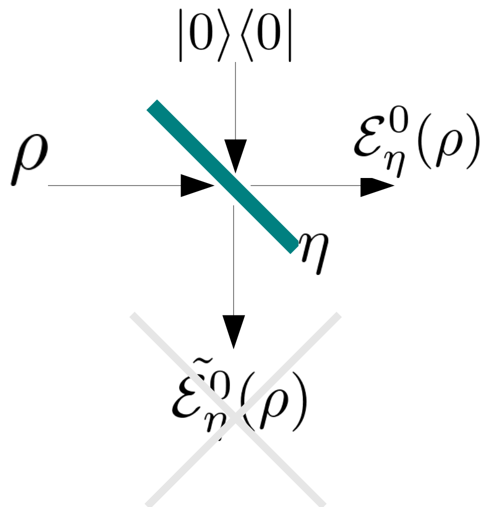
Most common channels are phase-insensitive: **quantum limited attenuator,**
thermal attenuator,
quantum limited amplifier,
thermal amplifier,
additive Gaussian noise.

Gaussian phase-insensitive channels

Quantum limited attenuator (beam splitter)

$$\mathcal{E}_\eta^0(\rho) = \mathrm{Tr}_{\mathrm{B}}[U(\rho \otimes |0\rangle\langle 0|)U^\dagger]$$

$$U^\dagger a U = \sqrt{\eta}\, a + \sqrt{1-\eta}\; b \qquad 0 < \eta < 1$$

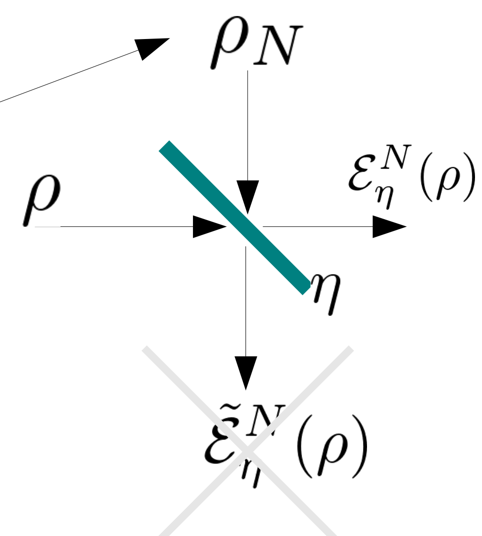


Thermal attenuator (thermal beam splitter)

$$\mathcal{E}_\kappa^N(\rho) = \mathrm{Tr}_{\mathrm{B}}[U\rho \otimes \rho_N U^\dagger]$$

thermal state with

$$N = \langle a^\dagger a\rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$$

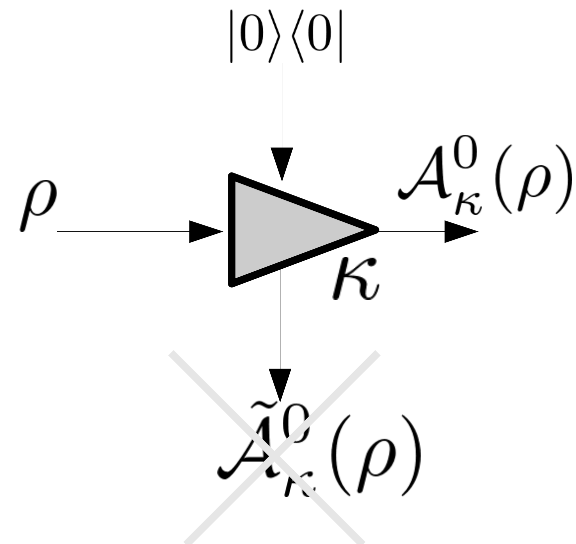


Gaussian phase-insensitive channels

Quantum limited amplifier

$$\mathcal{A}_\kappa^0 = \mathrm{Tr}_{\mathrm{B}}[U(\rho \otimes |0\rangle\langle 0|)U^\dagger]$$

$$U^\dagger a U = \sqrt{\kappa}a + \sqrt{1-\kappa}b^\dagger \qquad 1 < \kappa$$

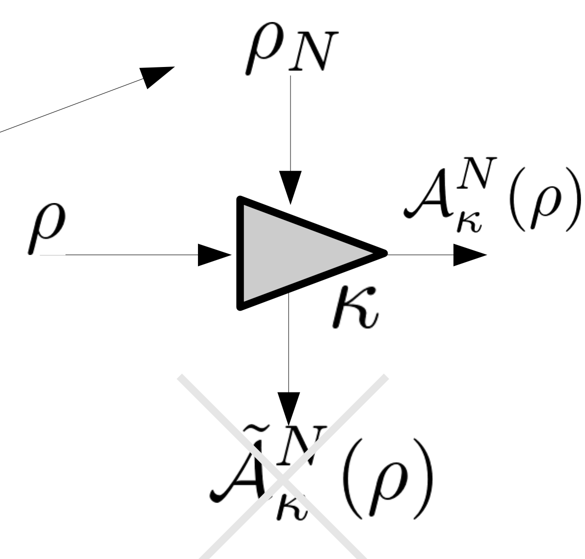


Thermal amplifier

$$\mathcal{A}_\kappa^N(\rho) = \mathrm{Tr}_{\mathrm{B}}[U\rho \otimes \rho_N U^\dagger]$$

thermal state with

$$N = \langle a^\dagger a\rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$$

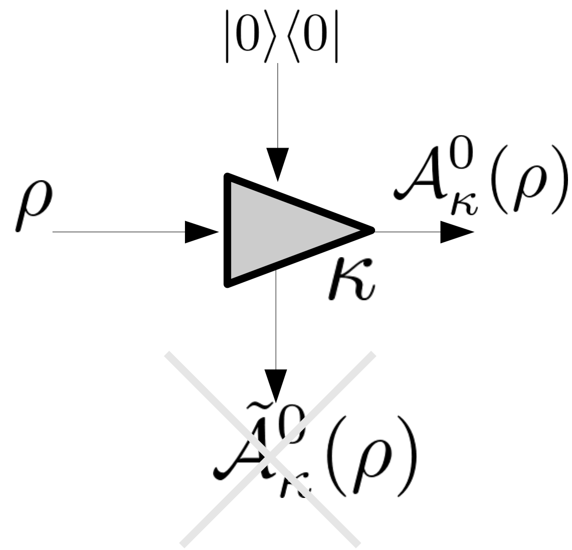


Gaussian phase-insensitive channels

Quantum limited amplifier

$$\mathcal{A}_\kappa^0 = \mathrm{Tr}_\mathrm{B}[U(\rho \otimes |0\rangle\langle 0|)U^\dagger]$$

$$U^\dagger a U = \sqrt{\kappa}\, a + \sqrt{1-\kappa}\, b^\dagger$$

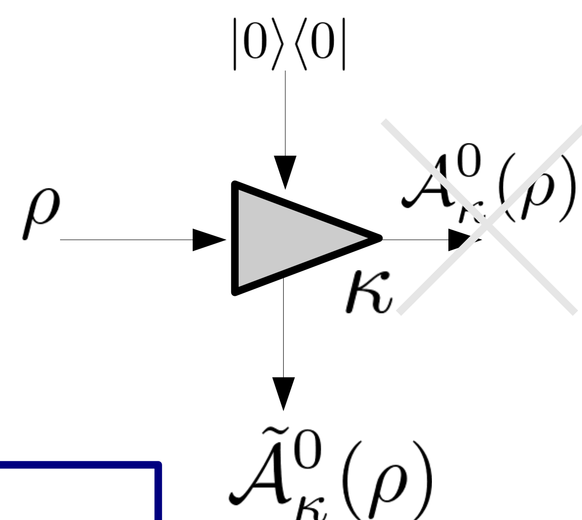


Complementary of quantum limited amplifier
(this is not phase-insensitive)

$$\tilde{\mathcal{A}}_\kappa^0 = \mathrm{Tr}_\mathrm{S}[U(\rho \otimes |0\rangle\langle 0|)U^\dagger]$$

phase-conjugation

$$U^\dagger b U = \sqrt{\kappa - 1}\, a^\dagger + \sqrt{\kappa}\; b$$



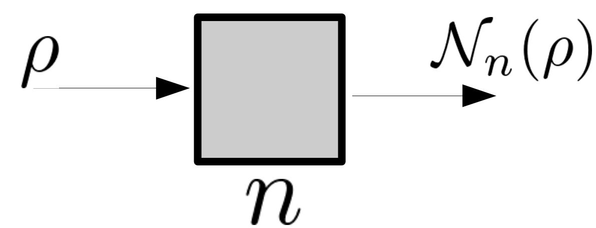
Important property: for a pure input $\rho = |\psi\rangle\langle\psi|$ the complementary output states $\mathcal{A}_\kappa^0(|\psi\rangle\langle\psi|)$ and $\tilde{\mathcal{A}}_\kappa^0(|\psi\rangle\langle\psi|)$ have the **same spectrum.**

Gaussian phase-insensitive channels

Additive classical noise channel

$$\mathcal{N}_n(\rho) = \int \frac{e^{-\frac{|\mu|^2}{2n}}}{2\pi n} D(\mu)\rho D^\dagger(\mu) d^2\mu$$

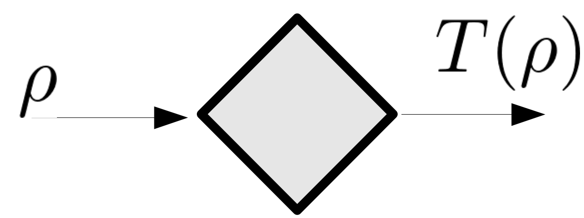
Random displacement with variance n



Noiseless phase conjugation
(this is not phase-insensitive and not CPT)

$$T(\rho) = \rho^\top$$

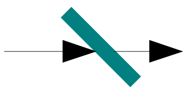
- Transposition in Fock basis
- Equivalent to time inversion



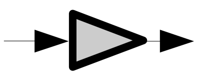
Important property: for any state ρ,
ρ and $T(\rho)$ have the **same spectrum.**

Gaussian phase-insensitive channels

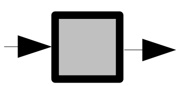
Summary in terms of characteristic functions



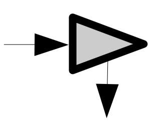
Attenuator $\chi(\mu) \rightarrow \chi(\sqrt{\eta}\mu)e^{-(1-\eta)(N+1/2)|\mu|^2}, \quad 0 < \eta < 1$



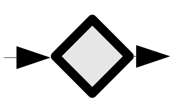
Amplifier $\chi(\mu) \rightarrow \chi(\sqrt{\kappa}\mu)e^{-(\kappa-1)(N+1/2)|\mu|^2}, \quad \kappa > 1$



Additive classical noise $\chi(\mu) \rightarrow \chi(\mu)e^{-n|\mu|^2}, \quad n > 0$



Complementary of q. lim. amplifier $\chi(\mu) \rightarrow \chi(-\sqrt{\kappa-1}\mu^*)e^{-\kappa\frac{|\mu|^2}{2}}, \quad \kappa > 1$
(phase-contravariant)



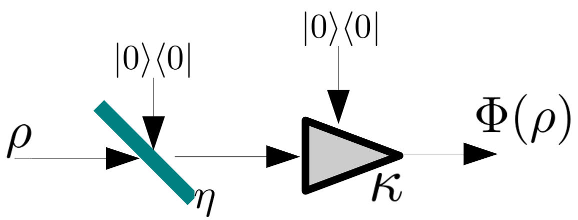
Noiseless phase-conjugation $\chi(\mu) \rightarrow \chi(-\mu^*)$
(phase-contravariant, not CPT)

Gaussian phase-insensitive channels

Giovannetti, *et al.*, PRA, (2004)
Garcia-Patron *et al.* PRL, (2012)

Every phase-insensitive Gaussian channel is equivalent to a **quantum limited attenuator** followed by a **quantum limited amplifier**.

$$\boxed{\Phi \equiv \mathcal{A}_\kappa^0 \circ \mathcal{E}_\eta^0}$$

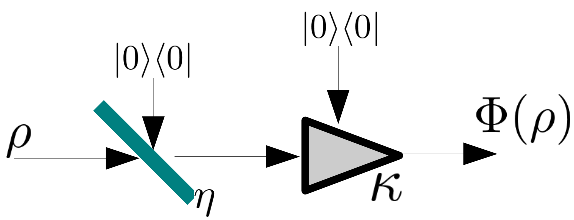


Gaussian phase-insensitive channels

Giovannetti, *et al.*, PRA, (2004)
Garcia-Patron *et al.* PRL, (2012)

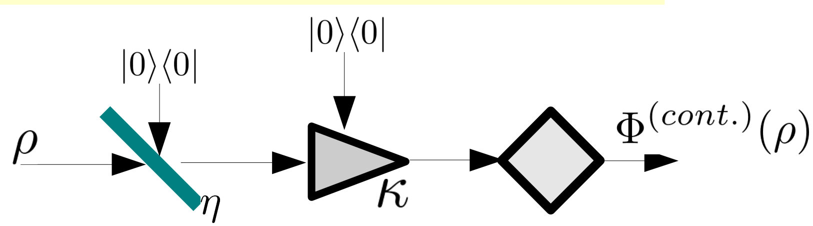
Every phase-insensitive Gaussian channel is equivalent to a **quantum limited attenuator** followed by a **quantum limited amplifier**.

$$\Phi \equiv \mathcal{A}_{\kappa}^{0} \circ \mathcal{E}_{\eta}^{0}$$

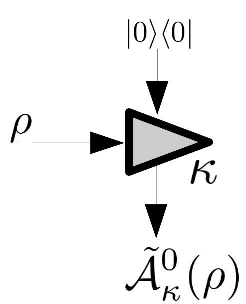


Every phase-contravaraint Gaussian channel is equivalent to a **quantum limited attenuator** followed by a **quantum limited amplifier,** and a **noiseless phase-conjugation**.

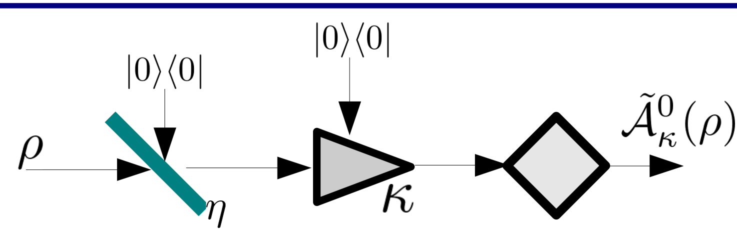
$$\Phi^{(cont.)}(\rho) = T \circ \mathcal{A}_{\kappa}^{0} \circ \mathcal{E}_{\eta}^{0}$$



Example: complementary of quantum limited amplifier



$\equiv$



$$\eta = 1 - 1/\kappa$$

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Outline

Long introduction

- General question behind this talk
- Overview of recent breakthroughs on the field
- Gaussian states and Gaussian channels
- **An old result about the quantum beam splitter**

Proof of the result

- Proof of the majorization conjecture
- Implications and outlook

An old result about the quantum beam splitter

ANNALS OF PHYSICS: **39**, 498–512 (1966)

A Quantum Characterization of Classical Radiation*

Y. AHARONOV,† D. FALKOFF, E. LERNER,‡ AND H. PENDLETON

Physics Department, Brandeis University, Waltham, Massachusetts

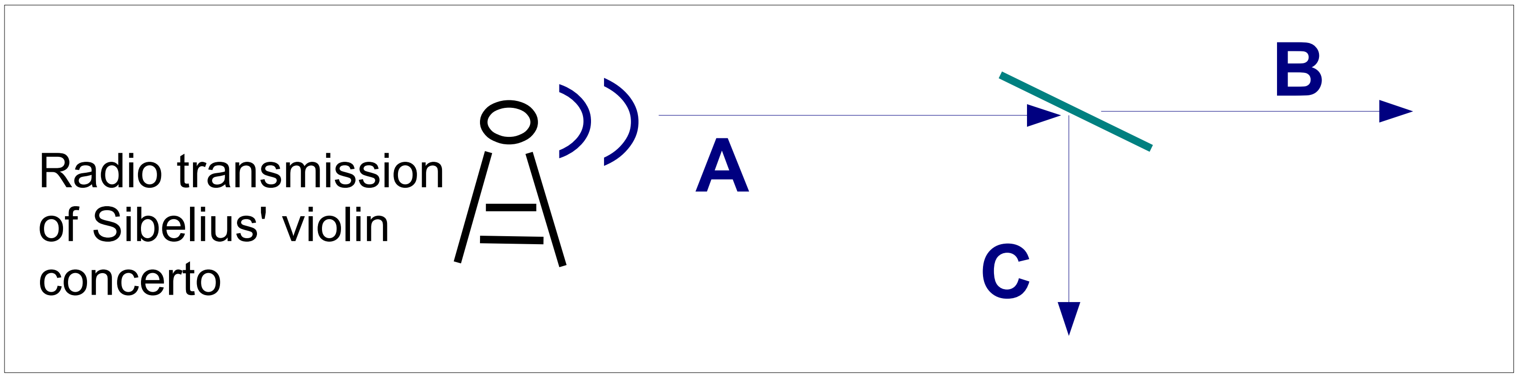
Any electromagnetic signal is representable, in the quantum mechanical description, by a suitable combination of photon states. We consider the question of which of the infinite number of possible combinations should correspond to a classical signal. The characteristic classical criterion we adopt is the indistinguishability of the radiation in two separate channels, whether it has been produced by independent sources or by a single source whose output is divided between the channels. For a quantum source a distinction is in general possible. We prove that the unique quantum state for which a distinction is not possible is the pure state characterized by Glauber as maximally coherent. The connection of this indistinguishability property with characteristic differences between classical and quantum measurements is emphasized.

Aharanov *et al.,* Ann. of Phys., (1966).

An old result about the quantum beam splitter

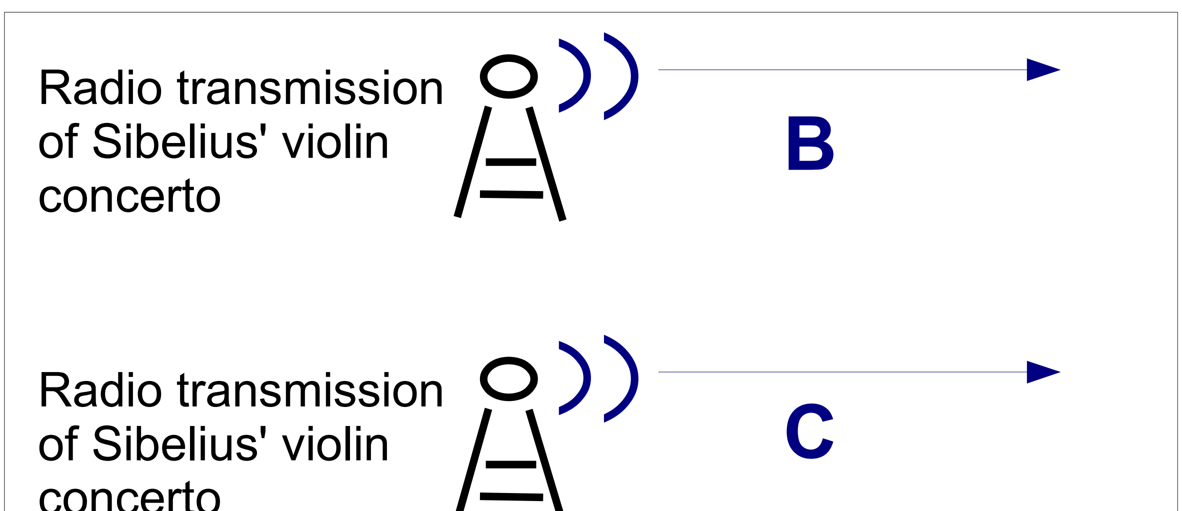
Case 1

the signal from A is divided between the circuits carrying it to B and C. Suppose the signal from A is the carrier modulated at audio frequencies by a recording of Sibelius' Violin Concerto.



Case 2

In case 2 the transmitters B and C are independent of each other; each has its own radio frequency oscillator and its own program of audio frequency modulation. Suppose that B and C both decide to play recordings of Sibelius' Violin Concerto; suppose the recordings are identical.



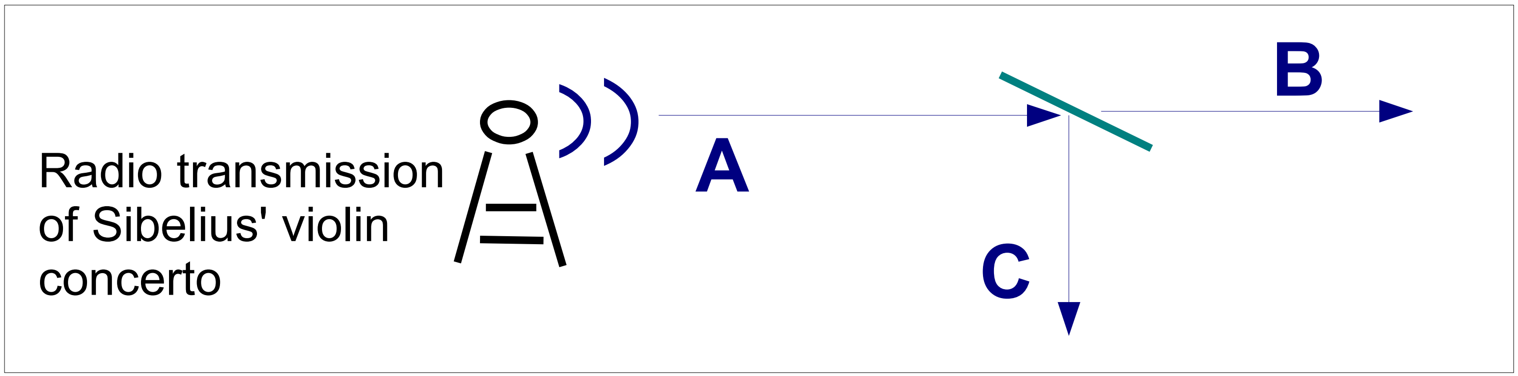
Is it possible to distinguish between case 1 and case 2 ?

Aharanov *et al.*, Ann. of Phys., (1966)

An old result about the quantum beam splitter

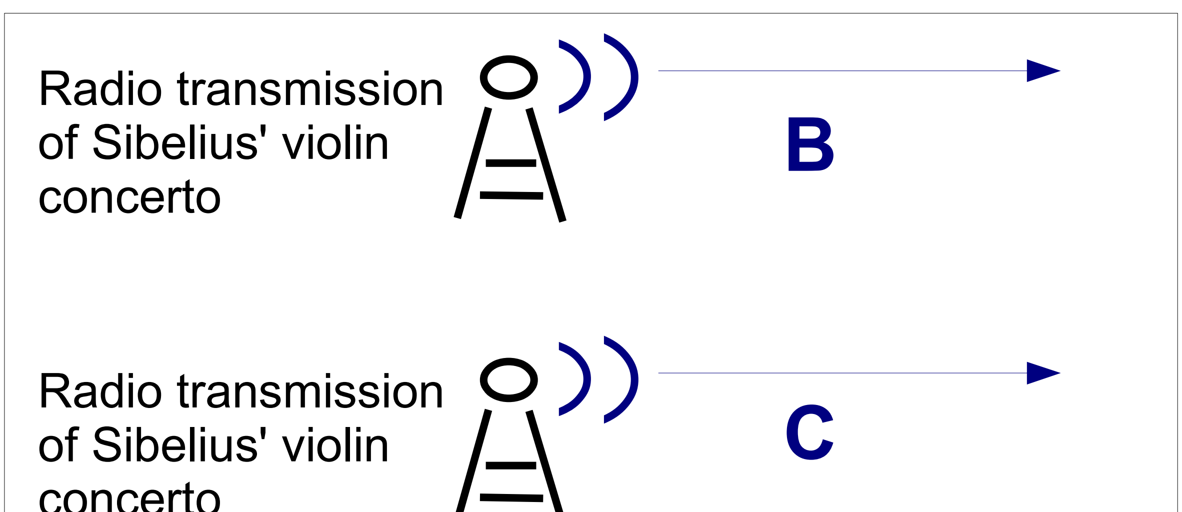
Case 1

the signal from A is divided between the circuits carrying it to B and C. Suppose the signal from A is the carrier modulated at audio frequencies by a recording of Sibelius' Violin Concerto.



Case 2

In case 2 the transmitters B and C are independent of each other; each has its own radio frequency oscillator and its own program of audio frequency modulation. Suppose that B and C both decide to play recordings of Sibelius' Violin Concerto; suppose the recordings are identical.



Is it possible to distinguish between case 1 and case 2 ?

It is always possible unless the transmitted signals are encoded on coherent states

Aharanov *et al.*, Ann. of Phys., (1966)

An old result about the quantum beam splitter

The "golden property" of a beam splitter*:*

the only input states producing pure output states for a quantum limited attenuator are coherent states.

Aharanov *et al.,* Ann. of Phys., (1966) Asbòth, Calsamiglia, Ritsch, PRL, (2005) Jiang, Lang, Caves, PRA, (2013)

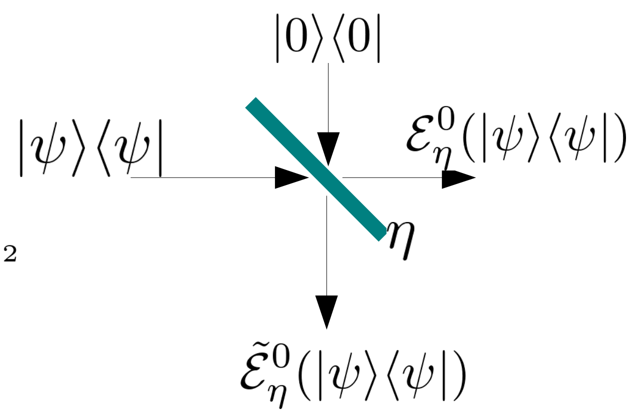
An old result about the quantum beam splitter

The “golden property” of a beam splitter*:*

the only input states producing pure output states for a quantum limited attenuator are coherent states.

Aharanov *et al.,* Ann. of Phys., (1966) Asbòth, Calsamiglia, Ritsch, PRL, (2005) Jiang, Lang, Caves, PRA, (2013)

proof: $\mathcal{E}_\eta^0[|\psi\rangle\langle\psi|] = |\psi'\rangle\langle\psi'|$



$$\tilde{\mathcal{E}}_\eta^0(|\psi\rangle\langle\psi|) = \mathcal{E}_{1-\eta}^0[|\psi\rangle\langle\psi|] = |\psi'_E\rangle\langle\psi'_E|$$

$$\chi'(z)\chi'_E(z_E) = \chi(\sqrt{\eta}z + \sqrt{(1-\eta)}z_E)e^{\frac{1}{2}|\sqrt{(1-\eta)}z - \sqrt{\eta}z_E|^2}$$

$$z_E = 0 \longrightarrow \chi'(z) = \chi(\sqrt{\eta}z)e^{\frac{1}{2}|\sqrt{(1-\eta)}z|^2}$$

$$z = 0 \longrightarrow \chi'_E(z_E) = \chi(\sqrt{(1-\eta)}z_E)e^{\frac{1}{2}|-\sqrt{\eta}z_E|^2}$$

$$\omega(z)\omega(z_E) = \omega(z + z_E)$$

$$\omega(z) \overset{\text{def}}{\equiv} \chi(z)e^{\frac{1}{2}|z|^2}$$

solutions are exp functions

$$\omega(z) = e^{z^*\alpha - z\alpha^*} \qquad \forall \alpha \in \mathbb{C}$$

$$\chi(z) = e^{-\frac{1}{2}|z|^2 + \bar{z}\alpha - z\bar{\alpha}}$$

characteristic function of coherent states

Outline

Long introduction

- General question behind this talk
- Overview of recent breakthroughs on the field
- Gaussian states and Gaussian channels
- An old result about the quantum beam splitter

Proof of the result

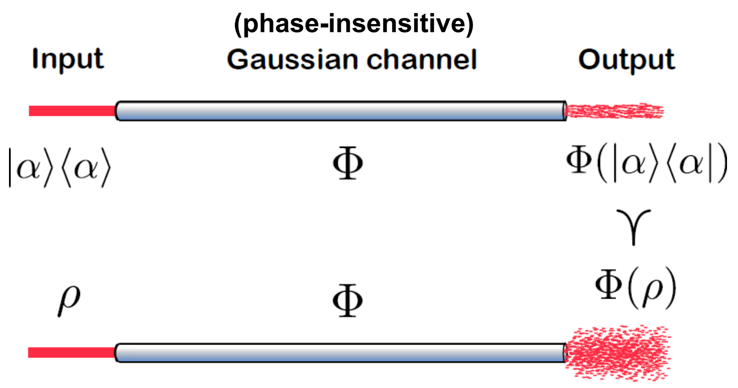
- **Proof of the majorization conjecture**
- Implications and outlook

Proof of the majorization conjecture

Majorization conjecture:

the output of coherent input states majorize all other output states

$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \quad \forall \rho, |\alpha\rangle$$

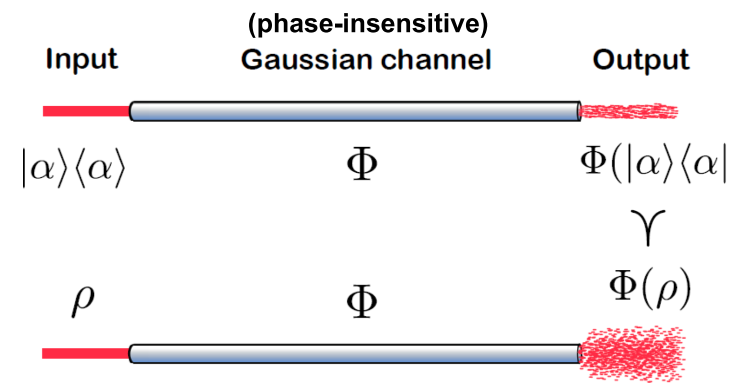


Proof of the majorization conjecture

Majorization conjecture:

the output of coherent input states majorize all other output states

$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \quad \forall \rho, |\alpha\rangle$$



We prove an equivalent and actually **stronger** proposition:

Minimization of stricly concave output funcitonals:

For every nonnegative, unitary invariant, and ***strictly concave*** *functional* F *and for every quantum state* ρ *,*

$$F(\Phi(\rho)) \geq F(\Phi(|\alpha\rangle\langle\alpha|)), \qquad \forall \alpha \in \mathbb{C},$$

moreover the equality sign is obtained **only if** ρ **is a coherent state.**

Strict concavity: $F(p\rho_1 + (1-p)\rho_2) \geq pF(\rho_1) + (1-p)F(\rho_2), \quad p \in (0,1)$

equality holds only if $\rho_1 = \rho_2$

Proof of the majorization conjecture

F is unitary invariant and strictly concave. Then the minimization problem $\min_{\rho} F(\rho)$ is trivially optimized by pure states and *only* by them.

Proof of the majorization conjecture

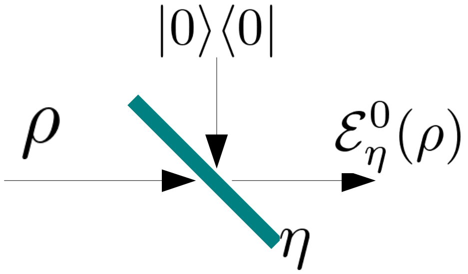
F is unitary invariant and strictly concave. Then the minimization problem $\min_\rho F(\rho)$ is trivially optimized by pure states and *only* by them.

Special case: quantum limited attenuator

$\mathcal{E}_\eta^0(\rho)$ is pure, if and only if, ρ is coherent.

"golden property"

Aharanov *et al.,* Ann. of Phys., (1966)



Then, $F(\mathcal{E}_\eta^0(\rho))$ is minimized ***only*** by coherent input states.

Proof of the majorization conjecture

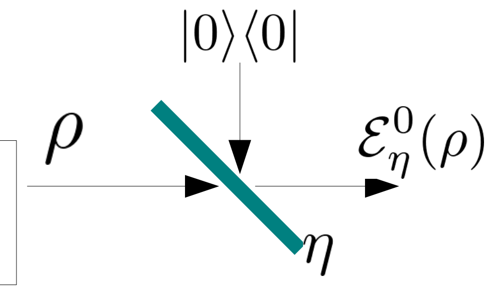
F is unitary invariant and strictly concave. Then the minimization problem $\min_{\rho} F(\rho)$ is trivially optimized by pure states and *only* by them.

Special case: quantum limited attenuator

$\mathcal{E}_\eta^0(\rho)$ is pure, if and only if, ρ is coherent.

"golden property"

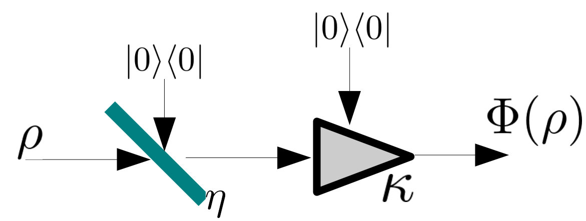
Aharanov *et al.*, Ann. of Phys., (1966)



Then, $F(\mathcal{E}_\eta^0(\rho))$ is minimized ***only*** by coherent input states.

General case: phase-invariant Gaussian channel

Decomposition property $\longrightarrow$ $\Phi \equiv \mathcal{A}_\kappa^0 \circ \mathcal{E}_\eta^0$



$\mathcal{E}_\eta^0$ maps coherent states into coherent states $\mathcal{E}_\eta^0(|\alpha\rangle\langle\alpha|) = |\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\alpha|$

It is enough to prove the theorem only for the quantum limited amplifier $\mathcal{A}_\kappa^0$

Garcia-Patron *et al.* PRL, (2012)

Proof of the majorization conjecture

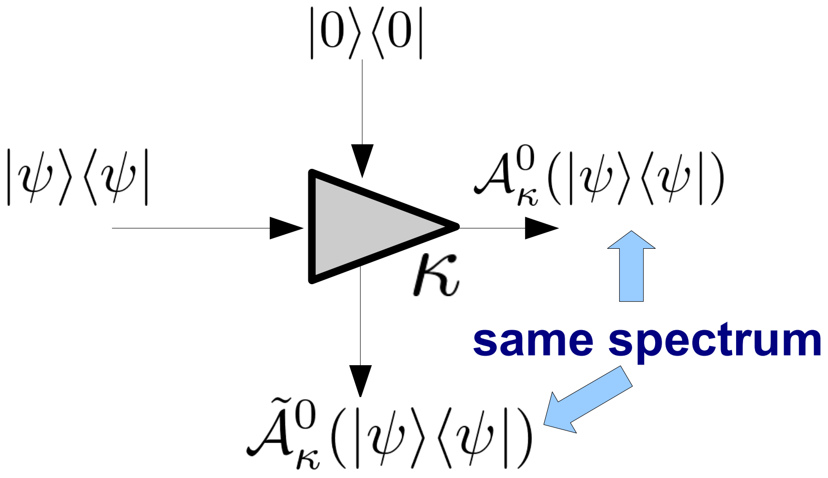
F is strictly concave $\longrightarrow$ it is minimized **only** by **pure** input states. (for “invertible” channels)

Proof of the majorization conjecture

F is strictly concave $\longrightarrow$ it is minimized **only** by **pure** input states. (for “invertible” channels)

Quantum limited amplifier applied to pure states

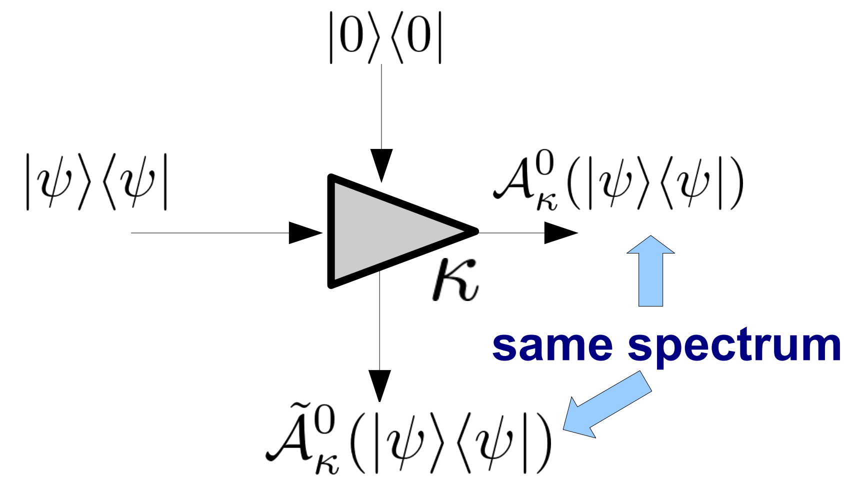
$$F(\mathcal{A}_\kappa^0(|\psi\rangle\langle\psi|) = F(\tilde{\mathcal{A}}_\kappa^0(|\psi\rangle\langle\psi|)$$



Proof of the majorization conjecture

F is strictly concave $\longrightarrow$ it is minimized **only** by **pure** input states. (for “invertible” channels)

Quantum limited amplifier applied to pure states



$$F(\mathcal{A}^0_\kappa(|\psi\rangle\langle\psi|) = F(\tilde{\mathcal{A}}^0_\kappa(|\psi\rangle\langle\psi|)$$

Decomposition of the complementary channel $\tilde{\mathcal{A}}^0_\kappa = T \circ \mathcal{A}^0_\kappa \circ \mathcal{E}_{\frac{\kappa-1}{\kappa}}$

$$F\left[\underline{\mathcal{A}^0_\kappa}(|\psi\rangle\langle\psi|)\right] = F\left[\underline{\mathcal{A}^0_\kappa} \circ \underline{\mathcal{E}^0_{\frac{\kappa-1}{\kappa}}(|\psi\rangle\langle\psi|)}\right]$$

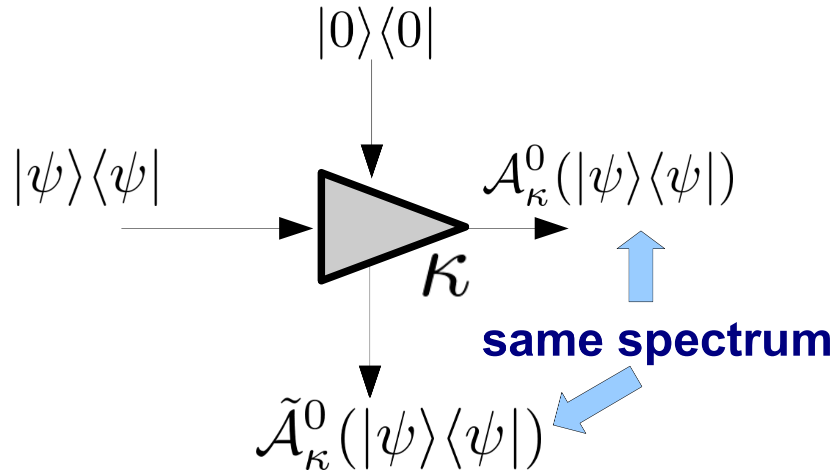
same channel !

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Proof of the majorization conjecture

F is strictly concave $\longrightarrow$ it is minimized **only** by **pure** input states. (for “invertible” channels)

Quantum limited amplifier applied to pure states



$$F(\mathcal{A}^0_\kappa(|\psi\rangle\langle\psi|) = F(\tilde{\mathcal{A}}^0_\kappa(|\psi\rangle\langle\psi|)$$

Decomposition of the complementary channel $\quad \tilde{\mathcal{A}}^0_\kappa = T \circ \mathcal{A}^0_\kappa \circ \mathcal{E}_{\frac{\kappa-1}{\kappa}}$

$$F\left[\mathcal{A}^0_\kappa(|\psi\rangle\langle\psi|)\right] = F\left[\mathcal{A}^0_\kappa \circ \mathcal{E}^0_{\frac{\kappa-1}{\kappa}}(|\psi\rangle\langle\psi|)\right]$$

same channel !

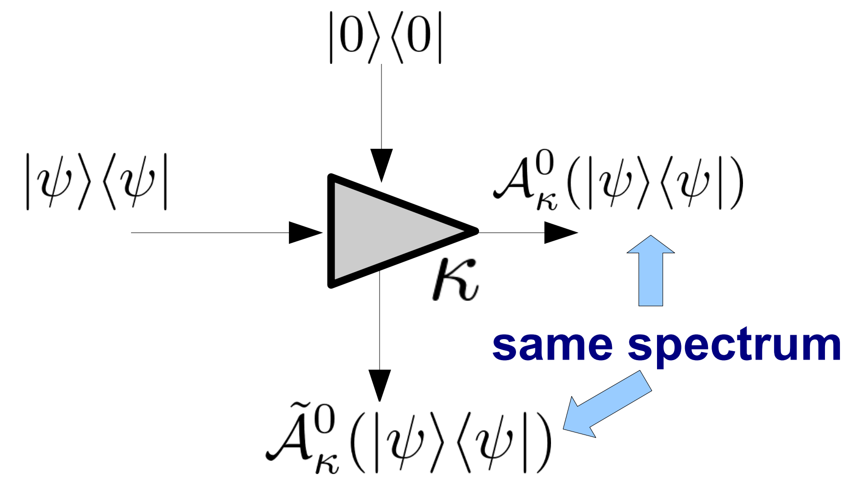
Giovannetti, Holevo, Garcia-Patron,
arXiv:1312.2251, Comm. Math. Phys., (2014)

up to this point same proof of

Proof of the majorization conjecture

F is strictly concave $\longrightarrow$ it is minimized **only** by **pure** input states. (for "invertible" channels)

Quantum limited amplifier applied to pure states



$$F(\mathcal{A}^0_\kappa(|\psi\rangle\langle\psi|) = F(\tilde{\mathcal{A}}^0_\kappa(|\psi\rangle\langle\psi|)$$

Decomposition of the complementary channel $\tilde{\mathcal{A}}^0_\kappa = T \circ \mathcal{A}^0_\kappa \circ \mathcal{E}_{\frac{\kappa-1}{\kappa}}$

$$F\left[\underline{\mathcal{A}^0_\kappa}(|\psi\rangle\langle\psi|)\right] = F\left[\underline{\mathcal{A}^0_\kappa} \circ \underline{\mathcal{E}^0_{\frac{\kappa-1}{\kappa}}(|\psi\rangle\langle\psi|)}\right]$$

same channel !

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

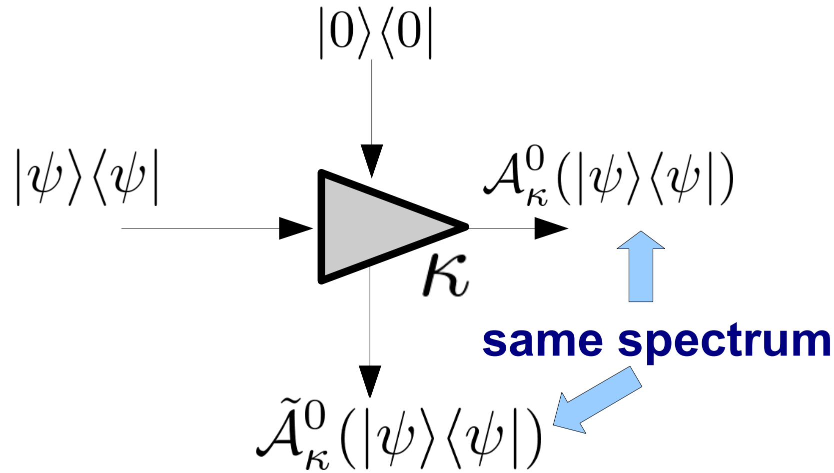
Take $|\psi\rangle\langle\psi|$ to be an optimal minimizer, then also $\mathcal{E}^0_{\frac{\kappa-1}{\kappa}}(|\psi\rangle\langle\psi|)$ must be optimal.

But optimal states must be pure $\longrightarrow$ $\mathcal{E}^0_{\frac{\kappa-1}{\kappa}}(|\psi\rangle\langle\psi|) = |\psi'\rangle\langle\psi'|$

Proof of the majorization conjecture

F is strictly concave $\longrightarrow$ it is minimized **only** by **pure** input states. (for "invertible" channels)

Quantum limited amplifier applied to pure states



$$F(\mathcal{A}^0_\kappa(|\psi\rangle\langle\psi|) = F(\tilde{\mathcal{A}}^0_\kappa(|\psi\rangle\langle\psi|)$$

Decomposition of the complementary channel $\tilde{\mathcal{A}}^0_\kappa = T \circ \mathcal{A}^0_\kappa \circ \mathcal{E}_{\frac{\kappa-1}{\kappa}}$

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

$$F\left[\underline{\mathcal{A}^0_\kappa}(|\psi\rangle\langle\psi|)\right] = F\left[\underline{\mathcal{A}^0_\kappa} \circ \underline{\mathcal{E}^0_{\frac{\kappa-1}{\kappa}}(|\psi\rangle\langle\psi|)}\right]$$

same channel !

Take $|\psi\rangle\langle\psi|$ to be an optimal minimizer, then also $\mathcal{E}^0_{\frac{\kappa-1}{\kappa}}(|\psi\rangle\langle\psi|)$ must be optimal.

But optimal states must be pure $\longrightarrow$ $\mathcal{E}^0_{\frac{\kappa-1}{\kappa}}(|\psi\rangle\langle\psi|) = |\psi'\rangle\langle\psi'|$

"golden property"

Aharanov *et al.*, Ann. of Phys., (1966)

this is possible if and only if $|\psi\rangle\langle\psi|$ is coherent. $\square$

We have just proved that:

Minimization of stricly concave output funcitonals:

*For every nonnegative, unitary invariant, and **strictly concave** functional* F *and for every quantum state* ρ *,*

$$F(\Phi(\rho)) \geq F(\Phi(|\alpha\rangle\langle\alpha|)), \qquad \forall \alpha \in \mathbb{C},$$

moreover the equality sign is obtained **only if** ρ is a coherent state.

We have just proved that:

Minimization of stricly concave output funcitonals:

For every nonnegative, unitary invariant, and ***strictly concave*** *functional* F *and for every quantum state* ρ *,*

$$F(\Phi(\rho)) \geq F(\Phi(|\alpha\rangle\langle\alpha|)), \qquad \forall \alpha \in \mathbb{C},$$

moreover the equality is sign is obtained **only if** ρ is a coherent state.

Majorization ~~conjecture~~:

the output of coherent input states majorize all other output states

$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \qquad \forall \rho, |\alpha\rangle$$

Outline

Long introduction

- General question behind this talk
- Overview of recent breakthroughs on the field
- Gaussian states and Gaussian channels
- An old result about the quantum beam splitter

Proof of the result

- Proof of the majorization conjecture
- **Implications and outlook**

We have just proved that:

Minimization of stricly concave output funcitonals:

*For every nonnegative, unitary invariant, and **strictly concave** functional* F *and for every quantum state* ρ *,*

$$F(\Phi(\rho)) \geq F(\Phi(|\alpha\rangle\langle\alpha|)), \qquad \forall \alpha \in \mathbb{C},$$

moreover the equality sign is obtained **only if** ρs a coherent state.

Majorization ~~conjecture~~:

the output of coherent input states majorize all other output states

$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \quad \forall \rho, |\alpha\rangle$$

We have just proved that:

Minimization of stricly concave output funcitonals:

*For every nonnegative, unitary invariant, and **strictly concave** functional* F *and for every quantum state* ρ *,*

$$F(\Phi(\rho)) \geq F(\Phi(|\alpha\rangle\langle\alpha|)), \qquad \forall \alpha \in \mathbb{C},$$

moreover the equality sign is obtained **only if** ρ s a coherent state.

$\Downarrow$ $\Downarrow$

Majorization conjecture:

the output of coherent input states majorize all other output states

$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \quad \forall \rho, |\alpha\rangle$$

(Strong) minimum output entropy conjecture:

*the output entropy is minimized **only** by coherent input states*

proof: take $F(\rho) = -\mathrm{Tr}[\rho \log(\rho)]$

We have just proved that:

Minimization of stricly concave output funcitonals:

*For every nonnegative, unitary invariant, and **strictly concave** functional* F *and for every quantum state* ρ ,

$$F(\Phi(\rho)) \geq F(\Phi(|\alpha\rangle\langle\alpha|)), \qquad \forall \alpha \in \mathbb{C},$$

moreover the equality sign is obtained **only if** ρ is a coherent state.

Majorization conjecture:

the output of coherent input states majorize all other output states

$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \quad \forall \rho, |\alpha\rangle$$

(Strong) minimum output entropy conjecture:

*the output entropy is minimized **only** by coherent input states*

proof: take $F(\rho) = -\mathrm{Tr}[\rho \log(\rho)]$

(Strong) minimum output Rényi entropy conjecture:

*the output Rényi entropy is minimized **only** by coherent input states*

proof: take $F(\rho) = 1 - \mathrm{Tr}[\rho^p]$

We have just proved that:

Minimization of stricly concave output funcitonals:

*For every nonnegative, unitary invariant, and **strictly concave** functional* F *and for every quantum state* ρ *,*

$$F(\Phi(\rho)) \geq F(\Phi(|\alpha\rangle\langle\alpha|)), \qquad \forall\alpha \in \mathbb{C},$$

moreover the equality sign is obtained **only if** ρ is a coherent state.

Majorization conjecture:

the output of coherent input states majorize all other output states

$$\Phi(|\alpha\rangle\langle\alpha|) \succ \Phi(\rho), \quad \forall\rho, |\alpha\rangle$$

(Strong) minimum output entropy conjecture:

*the output entropy is minimized **only** by coherent input states*

proof: take $F(\rho) = -\mathrm{Tr}[\rho\log(\rho)]$

(Strong) minimum output Rényi entropy conjecture:

*the output Rényi entropy is minimized **only** by coherent input states*

proof: take $F(\rho) = 1 - \mathrm{Tr}[\rho^p]$

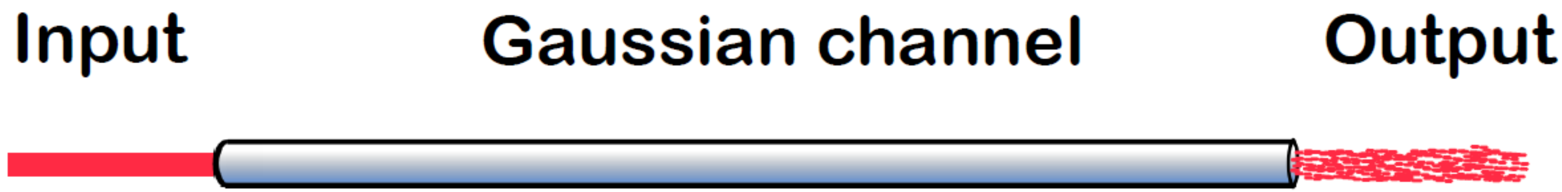
Phase space majorization

The (generalized) Q-function of a coherent state majorizes every other (generalized) Q-function

Lieb, Solovej, arXiv (2012)

proof: Giovannetti, Holevo, Mari arXiv:1405.4066

Conclusions



Radiation modes:
optical waves,
microwaves,
radio waves,
etc.

Physical transmission line:
optical fiber, free space communication,
satellite link, micro-waveguide, etc.

Noisy output signal

General question behind this talk:

“ What is the minimum “noise” or “disorder” achievable at the output state, optimizing over all possible input states? ”

Important implications in communication theory:
what are the code-words which are less disturbed by the channel ?

Conclusions

Input **Gaussian channel** **Output**

Radiation modes:
optical waves,
microwaves,
radio waves,
etc.

Physical transmission line:
optical fiber, free space communication,
satellite link, micro-waveguide, etc.

Noisy output signal

General question behind this talk:

“ What is the minimum “noise” or “disorder” achievable at the output state, optimizing over all possible input states? ”

Important implications in communication theory:
what are the code-words which are less disturbed by the channel ?

Answer: *Input coherent states produce the least “disordered” output states*

A hierarchy of ~~conjectures~~ proofs (state of the art)

Quantum Capacity

there are still gray problems...

Majorization conjecture

"the output of coherent input states majorize all other output states"

Mari, Giovannetti, Holevo, arXiv:1312.6225 Nat. Comm. (2014)

Giovannetti, *et al.*, PRA, (2004)

Phase-space majorization conjecture

"the Q-function of a coherent states majorizes every other Q-function"

V. Giovannetti, A. S. Holevo, A. Mari arXiv:1405.4066

Lieb, Solovej, arXiv:1208.3632

Entropy photon-number inequality conjecture

Guha, Shapiro, Erkmen, Proc. Inf. Th., IEEE, (2008).

Minimum output entropy conjecture

"the output entropy is minimized by coherent input states"

Giovannetti, Holevo, Garcia-Patron, arXiv:1312.2251, Comm. Math. Phys., (2014)

Giovannetti, *et al.*, PRA, (2004)

Additivity and minimum of output Rényi entropies conjecture

"output Rényi entropies of every order p>1 are additive and minimized by coherent input states"

Giovannetti, *et al.*, PRA, (2004)

Entropy power inequality conjecture

De Palma, Mari, Giovannetti, arXiv: 1402.0404, Nat. Phot. (2014)

König, Smith, Nat. Photon, (2013)

De Palma, Mari, Lloyd, Giovannetti, arXiv:1408.6410

bounds for classical capacity

Gaussian optimal encoding conjecture

"the max of classical capacity is obtained by Gaussian input states"

Gaussian additivity conjecture

"The Holevo information is additive (entangled code-words are not required)"

Holevo, Werner, PRA, (1997)

Strong converse theorem for classical capacity of Gaussian channels

Bardhan, Garcia-Patron, Wilde, Winter arXiv:1401.4161, IEEE Inf. Th., (2014)

CLASSICAL CAPACITY

(explicit formula for phase-insensitive Gaussian channels)

Giovannetti, Garcia-Patron, Cerf, Holevo arXiv:1312.6225, Nat. Phot. (2014)

Supplementary material

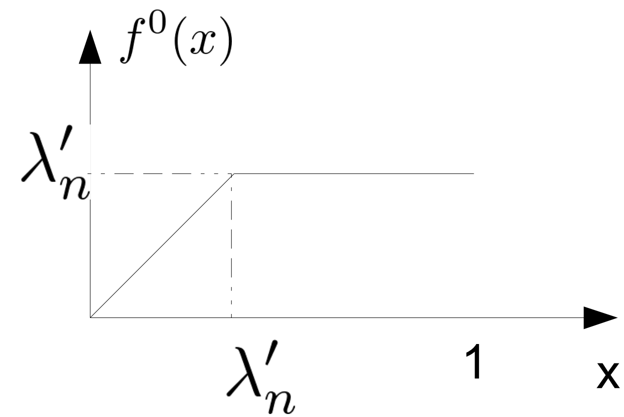
For every non-negative, strictly concave function f(x),

$$\lambda' \succ \lambda \quad \Leftrightarrow \quad \sum_j f(\lambda'_j) \leq \sum_j f(\lambda_j)$$

Proof ($\Longrightarrow$): follows trivially from concavity

Proof ($\Longleftarrow$): assume $\lambda' \nsucc \lambda$ then there exists a minimum integer n such that

$$\sum_{j=1}^{n} \lambda'_j < \sum_{j=1}^{n} \lambda_j$$



construct the function $f^0(x) := \begin{cases} x, & \text{if } 0 \leq x \leq \lambda'_n, \\ \lambda'_n, & \text{if } \lambda'_n \leq x \leq 1. \end{cases}$

violates the initial inequality $\sum_j [f^0(\lambda'_j) - f^0(\lambda_j)] = \delta > 0$

Problem: $f^0(x)$ is concave but not strictly concave.

ϵ small enough

It's not a big problem, just make it strictly concave: $f^\epsilon(x) := f^0(x) - \epsilon x^2$ $\square$